MATHEMATICAL MODEL FOR SIMULATION OF DAILY OUTFLOW FROM A KARSTIFIED AQUIFER

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Abstract: This paper describes structure of a mathematical model for simulation of daily discharge of a karstic spring. The model is based on the water balance of a karstified massif within the hydrologic cycle. The input vector consisting of meteorological components is known, while the system state (extractable aquifer capacity) and the output (yield of the spring) are unknown. It is assumed that the data on daily rainfall and average daily temperature are available for the considered long-time period. In order to provide for the model calibration, in addition to the size of the considered drainage area and its geological structure, it is necessary to collect a short-term measurements of daily discharge of the analyzed spring or of its «analog» within the considered long-time period. The results of the model simulation have been illustrated for several karstic springs in Serbia featuring different physical and geomorphologic characteristics.
Key words: mathematical model, simulation od daily outflow, karstic spring, karstic massif, extractable aquifer capacity

MATHEMATISCHES MODELL FUER MEHRJAEHRIGE SIMULIERUNG TAEGLICHER AUSFLUESSEN AUS KARSTQUELLEN

Zusammenfassung: In der Arbeit wurde die Struktur matematisches Modells fuer die Simulierung taeglicher Ausfluessen aus den Karstquellen dargestellt. Im Grund des Modells ist die Gleichung der Bilanz wasser - hydrologisches Zyklus des Karstmassivs, wo die Eingaenge (meteorologische Parameter) bekannt und die Wassermenge (dynamisches Volumen), die Ausgaenge (Ausfluss aus der Karstquelle) unbekannt sind. Under Voraussetzung, dass tagliche Summe des Neiderschlages und mitteltaeglicher Luftthemperatur im mehrjaehriger Periode bekannt sind. Fuer die Beduerfnisse des Parametertarierens des Modells muss man, neben der Flaeche des Zusammenflusses und ihrer Struktur auch mit den Messdaten taeglicher Fuelle aus derselben oder "analoger" Karstquelle in einer Periode innerhald betrachteter, mehrjaehringer Zeitspanne verfuegen. Die Ergebnisse der Anwendung ausgearbeitetes mathematisches Modells wurden fuer die Karstquelle Grza im Gebiet des Ostserbiens dargestellt.
Schlüsselworte: mathematisches Modell, Simulation taeglicher Ausfluessen, Karstquelle, Karstmassiv, dinamisches Volumen.

1. Introduction

The outflow from karstic springs is complex process within the development of the river runoff. The outflow complexity results from a very complex geological structure, the water is moving through on its way from precipitation to karstic spring outflow. With regard to it, it is reasonably to assume that karstic massifs, characterized by fractural porosity, comprise underground voids of larger (storages) and smaller size. The size, shape and character of these voids directly govern the groundwater flow through the considered porous environment. Under these circumstances, the ground water flow is difficult to explain from hydrological aspect. The outflow from karstic spring represents a “response” of the catchment to the precipitation, which can take various forms. The transformation of precipitation into the spring discharge can be very rapid and intensive, resulting in a steep and short discharge hydrograph. On the other hand, due to the size of underground voids (i.e. due to the aquifer retention capacity), the transformation could be very slow, producing small fluctuation of discharge within a long lasting hydrographs. Between these two boundary forms of discharge hydrographs a wide range of sub-variants and combinations is possible. In defining the form of hydrograph, the hydrometric measurement and observation of the regime proved to be the most reliable approach. However, in practice this is not always the case. The discharge is systematically observed on a small number of karstic springs. More often, the observations are sporadic, done for specific purpose, and performed during short time periods, or not performed at all.

The objective of investigation presented in this paper is to define a procedure for simulation of a long - term karstic spring discharge hydrograph, if short - term data for the same spring are available. The simulation is based upon the required meteorological events (precipitation and air temperature) prevailing in the karstic spring catchment area, as well as data on the run off of the catchment area “fed” by the analyzed spring, or data on the “analog” karstic spring discharge in a long term period.

2. Mathematical model structure

A mathematical model for simulation of karstic spring’s long-term daily outflow has been developed at several independent levels. Each level implies a certain number of functions, different in terms of mathematical structure and time increment, all performed with the aim to define daily discharges over a long time. All model input data are basically the same, so that the use of the same hydro-meteorological data base possible.

In this stage of development, there are five levels (modules) of the model relating different computation functions and purposes. The levels are:

- Level 1 - filling in missing data of the mean monthly discharge series applying VNC Model
- Level 2 - determining the length of design period for estimation of the elements of a karstic Aquifer long-term water balance - INTKR
- Level 3 - making water balancing of karstic springs - BALANCE
- Level 4 - identifying transformation functions module parameters -TRUNSFUNK
- Level 5 - simulation of the daily discharges over a long-term period - SIMIST

The pre-requisite for developing the model implies that the simulated series of a long-term daily discharge has to be determined in such a way as not to affect the harmony between the basic components of water balance established upon the monthly data basis.

2.1 Level 1 – Filling out missing data of the mean monthly discharge series using VNC model

The concept of Level 1 is to establish needed cause-effect relations based upon available long-term hydro- meteorological monthly values within a wider region based upon the available short-term data collected at karstic spring. The obtained relationships are used in the development of mathematical model for simulation of the mean monthly discharges of the analyzed spring within the considered longer period.

For filling in missing values and extending the mean monthly flow, series of the karstic spring the VNC model is used. Essentially, this model establishes linear correlation between the standardized series of mean monthly discharge and the corresponding climatic parameters governing the formation of outflow from karstic spring. The established correlation parameters are spatially analyzed and, in the regions where they prove to be homogenous, they are inversely used for simulation of the corresponding discharge series within a longer period, provided the corresponding data of the “analog” catchment and meteorological data of the analyzed catchment are available.

In the concrete case, using the VNC model (Prohaska et al, 1979) the inter-dependence of the standardized variables of the following hydro-meteorological values is expressed as:

$$U(Q_{ij}) = a_{01} \cdot U_1(Q_{ij}^a) + a_{02} \cdot U_2(P_{ij}) + a_{03} \cdot U_3(T_{ij}) + a_{04} \cdot U_4(V_{ij}) + a_{05} \cdot U_5(N_{ij}) \qquad (1)$$

Where:

$U_k(X_{ij})$ - Standardized value of the corresponding variable X_{ij}

Q_{ij} - Karstic spring mean monthly outflows

Q_{ij}^a - Mean monthly discharge value of the "analog" river

P_{ij} - Mean monthly rainfall at the analyzed karstic catchment

T_{ij} - Mean monthly air temperature at the analyzed karstic catchment

V_{ij} - Mean monthly air humidity within the analyzed karstic catchment

N_{ij} - Mean monthly vapor pressure at the analyzed karstic catchment

a_{0l} - The unknown parameters - regression coefficient

$j = 1,2,... 12$ Ordinal number of a month

$i = 1,2,... N$ Ordinal number of a year in a series

$l = 1,2,.m,..M$ Number of independent variables – regression equation (1)

The procedure for defining the model parameters has been in details described in the literature (Prohaska & Ristic, 2002). The basis for the computation of the model parameters are correlation coefficients between the values of all standardized variables (both dependent and independent) contained in equation 1. Based upon the values of the standardized correlation coefficients r_{lm} the linear regression coefficients a_{01}, a_{02}, ..., a_{0M} are obtained.

2.2 Level 2 – Determining the design period for estimation of the karstic aquifer long term water balance - IMTKRI

In assessing the average long - term water availability of the considered cathchment, including as well karstic springs as a whole, one starts from the assumption that the process of water formation in a karstic massif has a stochastic character. This practically means that, like with a majority of hydro-meteorological and hydro-geological events, a certain stochastic regularity is present in fluctuation of rainy (wet) and dry periods, including years. As a rule, hydro-meteorological and hydro-geological processes occurring in nature over a long - term period are cyclical i.e. stochastic. In assessing the average long term water balance of the area it is necessary to provide sufficiently long time series of discharge (outflow) which cover several hydrologic cycles (one cycle comprises one rainy and one dry period). This is made possible by applying the VNC model described above.

In hydrological practice, a so-called integral curve of module deviation is used for the estimation of the cyclical nature of hydro-meteorological phenomena and processes, which basically represents a cumulative line of the standardized variable.

$$f(t) = \sum_{i=1}^{t} \frac{K_i - 1}{C_v} \qquad (2)$$

Where:

K_i - Module coefficient for the i-th year

$$K_i = \frac{Q_i}{\overline{Q}} \qquad (3)$$

Q_i - Average annual discharge for the i-th year

$\overline{Q}$ - Average long - term discharge

C_v - Coefficient of variation of the average annual discharge series

A graphical presentation of integral line of deviation indicates the exchange of rainy and dry years. A positive increment of the function $f(t)$ characterizes a humid, while dry periods are indicated by the negative function increment. In defining long - term water resources of the area, i.e. the design length of calculated period for estimation of long - term water availability we adopted the length of the time comprising two or more full cycles. Otherwise, the longest series of available data is adopted.

2.3 Level 3 –Groundwater balance of karstic aquifer - BALANCE

The water balance of a karstic aquifer primarily depends on the input-output hydro-meteorological data and the volume of cavities and pores in a karstified massif, as well as its ability to store certain quantities of water, which are then subsequently discharged through karstic springs. The ability of a karstified massif to temporarily retain larger or smaller volume of water depends on the so-called extractable aquifer capacity – the space of cavities and pores in a karstified massif. The extractable aquifer capacity of a karstified massif directly affects the water balance. In addition, it indicates the level and the character of water availability of a karstified massif

Since a karstified massif can be considered a system that transforms precipitation into the karstic spring discharge, then the precipitation falling over the whole catchment area of a karstic spring is the input into the system.

The system output is actual evapotranspiration from the catchment area of karstic spring.

The extractable aquifer capacity, expressed through the volume of water in cavities and pores, directly depends on imbalance of the above stated input-output hydro-metereological values. It actually represents the amount of available water of the karstic aquifer.

The basic equation for computation of water balance of the karstic spring, with a monthly time increments can be expressed as:

$$P_{ij} = h_{ij} + E_{ij} + \left(V_{ij} - V_{i,j-1}\right) = h_{ij} + E_{ij} \pm \Delta_{ij} \qquad (4)$$

Where:

P_{ij} - Depth of monthly precipitation over the karstic catchment area

h_{ij} - Depth of mean monthly outflow from karstic spring

E_{ij} - Monthly amounts of effective (actual) evapotranspiration in karstic catchment

V_{ij} - Storage (water volume) in the analyzed karstic aquifer

Δ_{ij} - Change of storage of groundwater in the karstified massif.

Subscripts i =1, 2, …, N pertains to year, while j = 1, 2,…, 12 denotes month.

Taking into account the availability of data, the water balance equation (4), has two unknown parameters, that is E_{ij} and Δ_{ij}. Therefore, for the estimation of the “actual evapotranspiration” E_{ij} from the catchment a new boundary condition had to be introduced. This boundary condition implies estimation of the “actual evapotranspiration” applying the above equation based on both actual precipitation values in the catchment area and outflow from the spring under condition that the volume of groundwater (ground water storage) are equal at the beginning and at the end of the considered computing period.

For the first approximation of the actual monthly amount of evapotranspiration, the numerical values of daily amounts of potential evapotranspiration were obtained by the modified Thornthwait method and based on known values of mean daily air temperatures and actual sunlight duration, according to the formula:

$$PET_{ik} = \frac{0.4 \cdot N_{ij}}{3} \cdot \left(\frac{10 \cdot T_{ik}}{I_i} \right)^a \quad \text{(mm)} \qquad (5)$$

Where:

PET_{ik} - Potential daily amount of evapotranspiration

N_{ij} - Monthly sum of sunshine duration

T_{ik} - Mean daily air temperature

I_i - Annual heat index

a - Parameter in the function of heat index

Annual heat index is obtained as a sum of monthly values of reduced mean monthly air temperatures.

$$I_i = \sum_{j=1}^{12} \left(\frac{T_{ij}}{5} \right)^{1.514} \qquad (6)$$

Polynomial function a is expressed as:

$$a = 6.15 \cdot 10^{-7} \cdot I^3 - 7.71 \cdot 10^{-5} \cdot I^2 + 1.79 \cdot 10^{-2} \cdot I + 0.49 \qquad (7)$$

In obtaining daily amounts of actual “effective” evapotranspiration, the iterative approach to computing the balance (Equation 4) was applied, assuming that:

- The storage volume in karstified massifs at the beginning of the calculated design period equals to the one measured at its end, i.e.:

  $V_0 \cong V_K$

  Where zero and K denote the beginning, i.e. the end of the considered calculation period
- A non-linear distribution of the actual amounts of daily evapotranspiration, so that during the days of rainfall

  $E_{ik} = PET_{ik}$

  While the amount of actual daily precipitation in the subsequent days decreases according to the formula:

  $$E_{j(k+\tau)} = \Theta^{2\tau} \cdot PET_{ik} \qquad (8)$$

  Where:

  E_{ik} - Actual daily amount of evapotranspiration

  τ = 1, 2, 3, ..., m – lag time in days

Monthly amount of actual evapotranspiration is obtained based on the “adjusted” values of daily amounts of actual evapotranspiration, taking into account the above stated conditions and limits, expressed by a formula:

$$E_{ij} = \sum_{k=1}^{m_{j+i-1}} E_{ik} \qquad (9)$$

Where:

m_j - Number of days in the j-th month in series.

Given the above-determined value of the actual monthly evapotranspiration, the storage volume in a karstic aquifer is computed at the end of the j-th month following the equation:

$$V_{ij} = P_{ij} - h_{ij} - E_{ij} + V_{i,j-1} \quad (10)$$

Starting from the basic assumption that all hydrometeorological variables in the equation (10) are of stochastic nature, it follows that the state of the storage volume in a karstic aquifer a random variable as well, which can be modeled by its statistical parameters: mean value, variance, asymmetry, and higher order statistical moments. Besides, the storage of karstic aquifer as random variable could be expressed by a theoretical distribution function, i.e. certain theoretical values could be obtained for different levels of probability, i.e. return periods. Extreme values of the aquifer storage capacity indicate that there is a possibility of having natural water equalization and a possibility for quantitative definition of extractable aquifer capacity which is obtained from a difference between the maximum V_{max} and the minimum of the aquifer storage capacity in the specified balancing period, i.e.:

$$DZ = V_{max} - V_{min} \quad (11)$$

Where:

DZ - Extractable water volume from karstic aquifer.

2.4 Level 4 – Distribution – transformation module "rainfall - runoff"- TRNSFUMK

In developing a mathematical module for transformation of precipitation into discharge from karstic spring one starts from the assumption that karstic aquifer is an open system, the state of which changes in space and time. The general system's mathematical formulation may be expresses by a relation:

$$Y(t) = H(X(t), t) \quad (12)$$

Where:

t - Time

$X(t)$ - Input vector

$Y(t)$ - Output vector

H - System's transform function that defines transformation of input into output vectors

In the concrete situation, precipitation $P(x, y, t)$ represents the input values while the spring discharge $Q(t)$ is the output, which is distributed in time. Thus, the following expression can be written:

$$Q(t) = H(P(x, y, t)) \quad (13)$$

Where:

x and y - coordinates of the precipitation station

The transform function consists of a number of components depending on the number of transformations, which could be used to define the considered process. A complete process of transformation of rainfall into the outflow from karstic spring is mainly divided into three sub-processes. It is necessary, therefore, to determine the three functions that will be used to define the system.

A production function often called the function of "losses". Namely, total precipitation falling over a catchment area represents gross precipitation. A part of precipitation evaporates (evapotranspiration), a part is infiltrated into soil and only one part flows as surface runoff to the stream flow. A part of rainfall, which from the watershed (through either surface or underground) reaches the karstic spring, is a net or effective rainfall. Effective rainfall values are determined based on metereological components: daily amount of precipitation and temperature, or in some other way, as was implicitly done in the part 2.3 of this paper.

The transformation of effective precipitation into the discharge hydrograph of the karstic spring is described using two functions:

- Distribution, by which the water volume obtained on the basis of effective precipitation is distributed in time (elementary hydrograph) and
- Propagation, by which the elementary hydrograph is transformed into the hydrograph at the outlet of the karstic spring

The function of distribution and propagation is often replaced by a transformation function, what is done in this paper too.

For the concrete case, the transformation of total precipitation into the effective precipitation is made by means of the run off coefficient on a monthly basis:

$$\varphi_{ij} = \frac{h_{ij}}{P_{ij}} \quad (14)$$

$$h_{ij} = P_{ef_{ij}}$$

That is:

$$P_{ef_{ij}} = \varphi_{ij} \cdot P_{ij} \quad (15)$$

Where the run off depth "h" is the output of Level 1, adjusted with long term water balance through Level 4.

Distribution-transformation function, which converts the net precipitation in the hydrograph of the karstic spring, is defined by a relation:

$$Q(k) = F \cdot \sum P_{ef(ij)} \cdot TF_{(k-s+1)} \quad (16)$$

Where:

$Q(k)$ - Hydrograph ordinate in the k-day

$P_{ef}(j)$ - Effective precipitation in the j-th month

$TF_{(k-s+1)}$ - Transformation function at the moment k-s+1 (1/day)

F - Watershed area (km^2)

For the computation of the TRANSFUNK transformation function at the v-th moment the equation is:

$$TF_{(v)} = \frac{1}{\tau \cdot (n-1)!} \cdot \left(\frac{v}{\tau}\right)^{n-1} \cdot e^{-\frac{v}{\tau}} \quad (17)$$

Where:

τ and n - parameters, where τ has the dimension of time (day), while n is a non-dimensional parameter

2.5 Level 5 – Simulation of Daily outflow from karstic springs

A component of mathematical model – SIMIST model is structured in such a way to enable the simulation of daily discharges from karstic springs in a long-term period as defined at Level 2. It assumes the defined mathematical structure of Level 4, which is the rainfall-runoff model TRANSFUNK, with all parameters and limitations defined in advance. The assumption is that in addition to daily yields at the analyzed spring within the insufficiently long period, the required input data in the considered long-term period are available, as follows:

- average daily amount of precipitation in the watershed

- mean daily air temperature in the watershed
- monthly values of sunlight duration
- mean monthly air humidity/moisture in the watershed
- mean monthly water vapor pressure
- average monthly yield at the analog river/spring

Mathematical basis of the SIMIST model is defined by equations (16) and (17). The parameters of these equations are taken from Level 4 – that is from the TRANSFUNK model.

In order to provide for successful functioning of the SIMIST model, all other levels have to be previously defined and calibrated using available observation and measurement data in the watershed (or in broader area).

Actually, SIMIST model synthesizes all the functions of the developed modules (level), which is the basic framework for the simulation model of long-term karstic spring outflow.

3. Mathematical model for simulation of Grza karstic aquifer daily outflow

3.1 Introduction

Grza karstic spring is located in the watershed of the river Crnica, a tributary to the river Velika Morava. Approximate watershed surface area amounts to 37.25 km^2. The karstic spring discharge is monitored by a weir and a staff gauge. Data on daily outflow from karstic spring for the period 1972 – 1988 and 1988 – 1991 are available. There are no other data on the measurement of other hydro-metereological parameters in the Grza watershed.

The model determined the daily outflow from Grza karstic spring using data from the surrounding rainfall gauging stations: Donja Mutnica, Izvor and metereological stations Cuprija I Crni Vrh have. The Crnica River at the gauging station Paracin was used as the analog river. Simulation of daily outflow from Grza karstic spring was made for the period from 1961 to 2000.

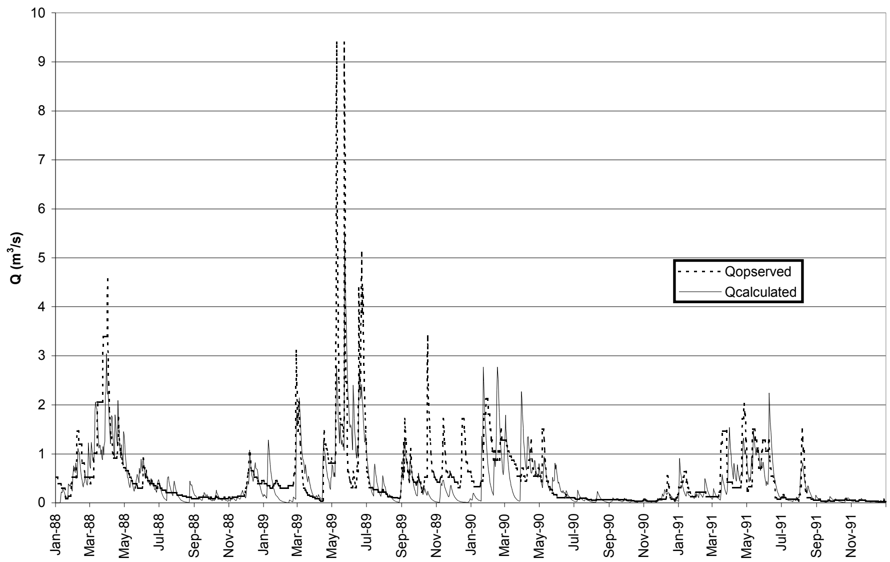
3.2 Review of the results computed

Calibration of the parameters of different levels (modules) of the mathematical model is performed successively. Using available data on the karstic spring outflow and the corresponding series of daily precipitation, applying the TRANSFUNK model, the τ and n parameters are determined first. Reliability and goodness of the results are illustrated in Figure 1 in a form of comparative diagraph of the realized and computed outflow in the period of observation and measurement. Then, at the level one, applying VNC model, long - term series of mean monthly outflow from Grza karstic spring is defined. The length of the design period for estimation of long term water availability of the analyzed karstic spring, and other components of water balance is determined by means of module 2 (level 2). Based on observed series of monthly precipitation in the watershed and the depth karstic spring flow, using module 3 (level 3), respecting the indicated conditions, the “actual” evapotranspiration in the watershed is determined. In this way, all elements for computing the water balance are obtained. The control of the results is made through the calculation of the karstic aquifer extractable water capacity at the end of the j-th month applying equation (10). The results of these computations can be seen in Figure 2. Based on the results shown in Figure 2 the overall extractable aquifer capacity of the Grza spring amounts to:

$$\Delta Z = 49.372 \cdot 10^6 m^3$$

This value indicates the significance of the Grza River drainage for the purposes of water management.

Table 1 illustrates the final comparative results of water balance for long - term observation period through average data of mean monthly and annual flow. The table also shows the percentage of monthly and annual deviations of the simulated discharge with respect to the observed values. Based on these indicators it could be concluded that monthly deviations range from -23.6% to +30.1 %. As for the mean annual values, the deviation is 1.6%. Relatively significant deviation of average mean monthly outflow is the result of



different characters (phases) of water capacity within the adopted design long term period.

Figure 1. Comparative hydrographs of numerical and observed discharge from Grza karstic spring in the period of observation and measurement

Table 1. Review of the mean monthly and annual discharge from the Grza spring for the period of observation and measurement and long-term calculated period

		Period of observation and measurement	Long term computing period	% deviation
	I	0.316	0.311	1.5
	II	0.562	0.486	13.5
	III	0.743	0.628	15.6
M	IV	0.631	0.780	-23.6
O	V	0.578	0.602	-4.2
N	VI	0.401	0.465	-16.0
T	VII	0.234	0.269	-14.9
H	VIII	0.184	0.157	14.6
S	IX	0.153	0.132	13.9
	X	0.166	0.116	30.1
	XI	0.148	0.122	17.5
	XII	0.227	0.206	9.1
GOD		0.362	0.356	1.6

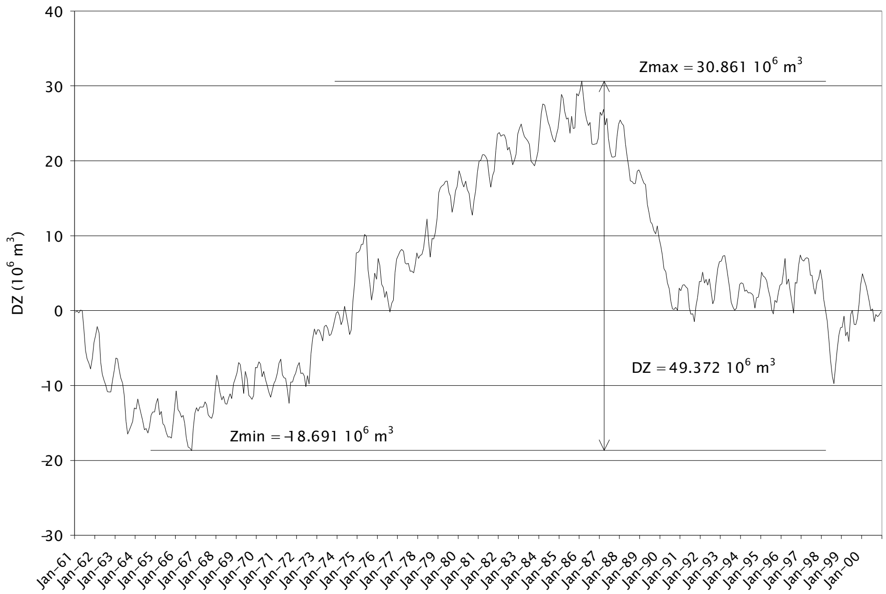


Figure 2. Water volume contained in the karstic aquifer of the Grza River for the period 1961 – 2000.

4. Conclusion

The analyzed mathematical model is suitable for determining all components of the long - term water balance in the karstic catchments. It is particularly suitable for estimation of the unknown actual evapotranspiration, which is most difficult to estimate at the catchments that have not been the subject of hydrologic investigations. In addition, the important model output is the series of daily discharges in a long-term period. Numerical value of total extractable karstic aquifer capacity represents significant product of the developed mathematical model.

Practical application of the model for the catchment of the Grze Spring showed presence of certain extractable aquifer capacity in the underground space, important for natural regulation of outflow from karstic massif. From the aspect of water balance, the total quantity of water that is discharged from the karstic spring over a long-term period is at average by 1.6 lower than the average discharge in the period of observation and measurement of the Grze karstic spring yield.

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